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LETTER TO THE EDITOR

Multi-soliton solutions in a finite depth fluid

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Abstract. A systematic procedure for solving the Whitham equation in a two-layer fluid of finite depth is developed. An analytic solution which asymptotically evolves into exactly two solitons is exhibited. The characteristics of these solitons can be quite different from those resulting from the Korteweg–de Vries equation (the shallow water limit of the present theory).

We consider the Whitham equation (Whitham 1967)

$$\frac{\partial u(x,t)}{\partial t}+Cu(x,t)\frac{\partial u(x,t)}{\partial x}+\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\mathrm{d}x'\,u(x',t)G(x'-x)=0,$$
$$G(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k\,c(k)\,\mathrm{e}^{ikx}, \qquad (1)$$

subject to the boundary conditions $u\to 0$ for $|x|\to\infty$, $c(k)$ being the infinitesimal-wave phase speed dispersion. For the 'thin thermocline model' (Phillips 1966) in a fluid of total depth D, thermocline located at the depth $z=-d$, the appropriate small-k form of $c(k)$ is

$$c(k)=c_0\left[1-\frac{1}{2}kd\left(\coth(kD)-\frac{1}{kD}\right)\right]. \qquad (2)$$

If $D\to 0$, $c(k)-c_0\sim k^2$ and equation (1) reduces to the Korteweg–de Vries equation (Benjamin 1966) whereas for $D\to\infty$, $c(k)-c_0\sim|k|$, and equation (1) reduces to the Benjamin–Ono equation (Benjamin 1967, Ono 1975). Substitution of equation (2) into equation (1) reduces it to the form

$$\frac{\partial u(x,t)}{\partial t}+c_0\frac{\partial u(x,t)}{\partial x}+Cu(x,t)\frac{\partial u(x,t)}{\partial x}+\frac{c_0d}{2D}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\mathrm{d}x'\,u(x',t)H(x'-x)=0,$$
$$H(x)=\mathrm{sgn}(x)[\exp(\pi|x|/D)-1]^{-1}.$$

It has recently been shown (Joseph 1977) that an exact stationary wave solution to equation (3) is given by

$$u(x-ct)=(c_0\gamma d/CD)\sin(\gamma)\{\cosh[\gamma(x-ct)/D]+\cos(\gamma)\}^{-1} \qquad (4)$$

where $c/c_0=1+(d/2D)[1-\gamma\cot(\gamma)]$, γ being an arbitrary real parameter with $0<\gamma<\pi$. The purpose of the present Letter is to outline a general method for obtaining particular non-stationary solutions to equation (3).

The term $c_0\partial u/\partial x$ in equation (3) can be transformed away by setting $x = \bar{x} + c_0 \ldots$, $t = \ldots$ We shall assume this to be done but still retain the (x, t) notation rather than use $(\bar{x}, \ldots)$, taking into account any differences in the final results. Note that if $u(x, t)$ is a solution, then so is $u(-x, -t)$. We represent the solutions in the following complex form:

$$u(x, t) = \sum_{n=0}^{\infty} a_n(t) \exp(-\mathrm{i}n\pi\lambda x/D), \tag{5}$$

λ being an arbitrary real parameter. Substitution of equation (5) into equation (3) leads directly to the following conditions on the $a_n(t)$:

$$\begin{aligned} &\dot{a}_0(t) = 0, \\ &\dot{a}_n(t) - Pf_n a_n(t) = \tfrac{1}{2} nQ \sum_{m=0}^{n} a_m(t) a_{n-m}(t), \qquad n > 0 \end{aligned} \tag{6}$$

where

$$f_n = n^2\left(\frac{1}{\gamma n} - \cot(\gamma n)\right) \tag{7}$$

with $P = \frac{1}{2} c_0 d(\gamma/D)^2$, $Q = C(\gamma/D)$ and $\gamma \equiv \pi \mathrm{i}\lambda$. These are to be solved subject to initial conditions $a_n(0)$ which are obtained from an assumed $u(x, 0)$ via equation (5). Since $\dot{a}_0(t) = 0$, $a_0(t) = \text{constant}$. We now restrict our attention to solutions for which this constant is set equal to zero. The lower and upper summation limits in equation (6) are then replaced by 1 and $n-1$, respectively. Hence the problem is reduced to solving a set of ordinary linear first-order recursive differential equations.

It is trivial to formally integrate equation (6). The solutions will in general be of the form

$$a_n(t) = \sum_{c^{(n)}} a(c^{(n)})\, \mathrm{e}^{Pf(c^{(n)})t} \tag{8}$$

where the summation is to be taken over all integer combinations consistent with the constraints

$$n = \sum_{j=1}^{n} jr_j, \qquad r_j \text{ a non-negative integer} \tag{9}$$

and

$$f(c^{(n)}) = \sum f_j r_j. \tag{10}$$

The core of finding the solution is a determination of the constants $a(c^{(n)})$, these being constrained by

$$a_n(0) = \sum_{c^{(n)}} a(c^{(n)}). \tag{11}$$

It can be shown that if we pick the $a_n(0)$, $n \leq N$, properly, while for $n > N$ have them satisfy equation (11) automatically, solutions result for which the summations can be exactly performed. To illustrate this we shall first recover the stationary wave solution given by equation (4). Consider solutions to equation (6) of the form

$$a_n(t) = a(n)\, \mathrm{e}^{nPf_1 t}, \qquad a(n) = a_n(0). \tag{12}$$

This $a_n(t)$ will solve equation (6) if the $a(n)$ satisfy the equation

$$\Delta(n)a(n) = \tfrac{1}{2}n\sigma \sum_{m=1}^{n-1} a(m)a(n-m) \tag{13}$$

where $\sigma \equiv Q/P$ and

$$\Delta(n) \equiv nf_1 - f_n = n[n \cot(n\gamma) - \cot(\gamma)]. \tag{14}$$

We take as the initial condition $a_1(0) = (4/\sigma)\sin(\gamma)$. It is then straightforward to verify that solutions to equation (13) are

$$a(n) = (4/\sigma)(-1)^{n+1} \sin(n\gamma). \tag{15}$$

Substitution of this $a(n)$ into equation (12) and whence into equation (5) gives

$$u(x, t) = (4/\sigma) \sum_{n=1}^{\infty} (-1)^{n+1} \sin(n\gamma)\, \mathrm{e}^{-n\xi} = (2/\sigma)\sin(\gamma)[\cosh(\xi) + \cos(\gamma)]^{-1} \tag{16}$$

where $\xi = (\gamma x/D) - Pf_1 t$. For λ real, γ pure imaginary, equation (16) represents a periodic solution. This function is bounded in x, t and λ. We can analytically continue it into the whole γ-plane, so that γ becomes a general complex number. In particular, this function will exist and be bounded for γ purely real, $0 < \gamma < \pi$ which we assume to be the case. Transforming back to the original reference frame shows this result to agree exactly with equation (4). Note that although this function is defined for all x, it cannot be formally expanded as a single series of exponentials which simultaneously converges for $x > 0$ and $x < 0$.

As a less trivial example, we now consider solutions which can evolve asymptotically into exactly two solitons. We write $a_n(t)$ in the form

$$a_n(t) = \sum_{q=1}^{M} a(q, n)\, \mathrm{e}^{Pf(q,n)t} \tag{17}$$

where $M = [\frac{1}{2}(n+2)]$ and

$$f(q, n) = (q-1)f_2 + (n+2-2q)f_1. \tag{18}$$

This $a_n(t)$ solves equation (6) if the $a(q, n)$ satisfy

$$\Delta(q, n)\hat{a}(q, n) = \tfrac{1}{2}n\sigma \sum_{m=1}^{n-1} \sum_{s=1}^{q} \hat{a}(s, m)\hat{a}(q+1-s, n-m) \tag{19}$$

where

$$\hat{a}(q, n) = \begin{cases} a(q, n) & 1 \leq q \leq M \\ 0 & q > M \end{cases} \tag{20}$$

and

$$\Delta(q, n) = 2(q-1)\tan(\gamma) + \Delta(n). \tag{21}$$

Picking $a_1(0) = (4/\sigma)\sin(\gamma)$, $a_2(0) = (4/\sigma)\sin(2\gamma)$, solutions to equation (19) can be shown to be

$$a(1, n) = (4/\sigma)(-1)^{n+1} \sin(n\gamma)$$

$$a(q, 2q-2) = (8/\sigma)(-1)^{q} \sin[2(q-1)\gamma], \qquad q > 1$$

$$a(q,n)=(4/\sigma)nB(-1)^{n+q}\sin(n\gamma)\sum_{k=0}^{q-2}\frac{B^k}{k!(k+1)!}$$

$$\times\prod_{j=0}^{k-1}{}'(n+1-2q-j)(q-2-j),\qquad q\geq 2, n>2q-2, \tag{23}$$

with

$$B=8\cos^2(\gamma)[1+8\cos^2(\gamma)]^{-1} \tag{23}$$

and the prime on Π means set the product equal to unity for $k=0$. Substituting equation (22) into equation (17) and whence into equation (5) then gives

$$u(x,t)=u_1+u_2+u_3 \tag{24}$$

where

$$u_1=\sum_{s=1}^{\infty}a(1,s)\,e^{-s\xi_1}=(2/\sigma)\sin(\gamma)[\cosh(\xi_1)+\cos(\gamma)]^{-1}$$

$$u_2=\sum_{s=1}^{\infty}a(s+1,2s)\,e^{-2s\xi_2}=(4/\sigma)\sin(2\gamma)[\cosh(2\xi_2)+\cos(2\gamma)]^{-1} \tag{25}$$

$$u_3=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}a(s+1,r+2s)\,e^{-r\xi_1}\,e^{-2s\xi_2}=\frac{4BJ}{\sigma K}$$

with

$$\begin{aligned}&\xi_1=(\gamma x/D)-Pf_1t,\qquad \xi_2=(\gamma x/D)-\tfrac{1}{2}Pf_2t,\\ &J=S[U(U-B)-V^2]-RV(2U-B)\\ &K=(U^2+V^2)[(U-B)^2+V^2]\end{aligned} \tag{26}$$

and

$$\begin{aligned}&U=1+z_1\cos(\gamma)+z_2\cos(2\gamma)+z_1z_2\cos(3\gamma)\\ &V=z_1\sin(\gamma)+z_2\sin(2\gamma)+z_1z_2\sin(3\gamma)\\ &R=z_1\cos(\gamma)+2z_2\cos(2\gamma)+3z_1z_2\cos(3\gamma)\\ &S=z_1\sin(\gamma)+2z_2\sin(2\gamma)+3z_1z_2\sin(3\gamma)\\ &z_1=\exp(\xi_1),\qquad z_2=\exp(2\xi_2).\end{aligned} \tag{27}$$

For γ pure imaginary, this solution represents the interaction of two complex periodic waves. Using the same kind of arguments as before, we can however take γ to be purely real with $0<\gamma<\frac{1}{2}\pi$. To show that this solution then asymptotically evolves into two solitons, we make the following identifications (γ real):

$$\begin{aligned}&\hat{c}_1=DPf_1/\gamma=(c_0d/2D)[1-\gamma\cot(\gamma)]\\ &\hat{c}_2=DPf_2/2\gamma=(c_0d/2D)[1-2\gamma\cot(2\gamma)]\end{aligned} \tag{28}$$

so that $\xi_1=(\gamma/D)(x-\hat{c}_1t)$, $\xi_2=(\gamma/D)(x-\hat{c}_2t)$. Now consider the limit $t\to+\infty$. Suppose that ξ_1 remains finite, so that $\xi_2\sim(\hat{c}_1-\hat{c}_2)t$. If ξ_2 remains finite, then $\xi_1\sim(\hat{c}_2-\hat{c}_1)t$. For $\hat{c}_1>\hat{c}_2$ and ξ_1 finite ($\xi_2\sim+\infty$) we find from equations (25)–(27)

$$u\sim(2/\sigma)\sin(\gamma)[\cosh(\xi_1)+\cos(\gamma)]^{-1} \tag{29}$$

while for ξ_2 finite ($\xi_2 \sim -\infty$),

$$u \sim (4/\sigma)\sin(2\gamma)[\cosh(2\xi_2+\mu)+\cos(2\gamma)]^{-1} \tag{30}$$

with $\mu = \ln[1+8\cos^2(\gamma)]$. Similarly for $\hat{c}_1 < \hat{c}_2$, for ξ_1 finite ($\xi_2 \sim -\infty$) we get a result of the form of equation (29) but with $\xi_1 \to \xi_1 + \mu$ while for ξ_2 finite ($\xi_1 \sim +\infty$) we get equation (30) but with $2\xi_2 + \mu \to 2\xi_2$. Hence aside from a phase shift (μ), this solution asymptotically evolves into two localised waves, each of the same shape as the stationary wave solution, equation (4), moving with constant speeds $c_1 = c_0 + \hat{c}_1$, $c_2 = c_0 + \hat{c}_2$ in the original reference frame. To show that these localised waves represent solitons, consider the limit $t \to -\infty$. The asymptotic forms for $c_1 > c_2$ are then given by those for the case $c_1 < c_2$, $t \to +\infty$ and *vice versa*, verifying that except for phase shifts, the two limits give the same results.

Since the two-soliton solution we have obtained depends on a parameter γ we in fact have a whole family of possible solutions, corresponding to the particular initial function $u(x, 0)$ characterised by a given value of γ. In general $u(x, 0)$ is not a symmetrical function of x, e.g. $u(x, 0) \neq u(-x, 0)$. A second set of independent solutions can be obtained from those given by the simple replacements $x \to -x$, $t \to -t$. It is interesting to compare the detailed form of the solitons with those of the Korteweg–de Vries equation. In the latter case, the peak soliton amplitudes (u_p) and speeds relative to $c_0 (c - c_0)$ are both proportional to the square of an appropriate real eigenvalue (κ_m) of $u(x, 0)$ (Gardner *et al* 1974). Hence both solitons would have peak asymptotic amplitudes of the same sign and must both move off in the same direction. Moreover, the ratio $(c - c_0)/u_p$ must be the same for these solitons. For the finite depth situation, the latter restriction is no longer true. To illustrate these points, figure 1 shows the spatial form of the solution at various times for the case $D/d = 5$ and $\gamma = 1{\cdot}4$. The asymptotic appearance of two localised parts is clear. Both solitons move off to the right and have peak amplitudes of the same sign. However, here $(c_2 - c_0)/(c_1 - c_0) = 11{\cdot}7$, while $u_{p2}/u_{p1} = 13.8$. This figure is drawn in the original frame of reference.

The generalisation of these results to an N-soliton case appears clear. For $n \leq N$ one can choose the $a_n(0)$ by $a_n(0) = (4/\sigma)\sin(n\gamma)$ and for $n > N$ by equation (11).

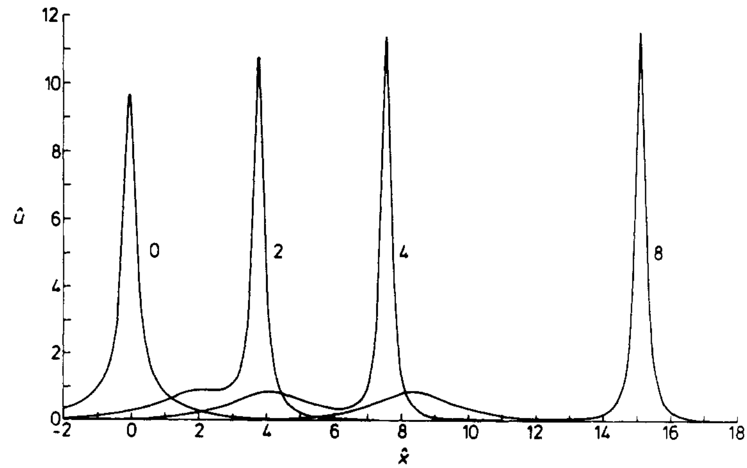

Figure 1. Dependence of reduced two-soliton solution $\hat{u}(=CDU/c_0\gamma d)$ on reduced spatial position $\hat{x}(=\gamma x/D)$ for various values of reduced time $\hat{t}(=\gamma c_0 t/D)$ for the case $D/d = 5$, $\gamma = 1{\cdot}4$. See equations (24)–(27).

Exactly N solitons asymptotically evolve in the form

$$u \sim \sum_{n=1}^{N} \frac{A_n}{\cosh(n\xi_n + \mu_n) + \cos(n\gamma)} \tag{31}$$

where $A_n = (2/\sigma)n\sin(n\gamma)$, $\xi_n = (\gamma/D)(x - c_n t)$, $c_n/c_0 = 1 + (d/2D)[1 - n\gamma\cot(n\gamma)]$, μ_n is some phase shift and γ is a real parameter, $0 < \gamma < \pi/N$.

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